

Closing Today: HW\_1A,1B,1C  
 Closing next Wed: HW\_2A,2B,2C  
 Read 5.3, 5.4 and 5.5 of the book.

### 5.3 The Fundamental Theorem of Calculus (FTOC)

Motivational Entry Task:

Consider the function  $f(t) = 3t$ .

Draw the graph and use area formulas you know, to compute:

$$1. \int_0^1 f(t) dt = \frac{1}{2} (1)(3) = \frac{3}{2}$$

$$2. \int_0^{10} f(t) dt = \frac{1}{2} (10)(30) = 150$$

$$3. g(x) = \int_0^x f(t) dt = \frac{1}{2} (x)(3x) = \frac{3}{2} x^2$$

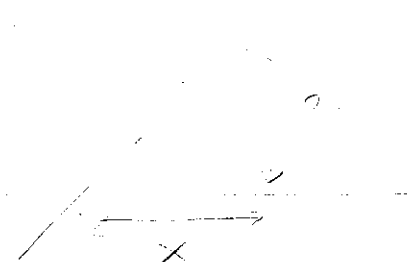
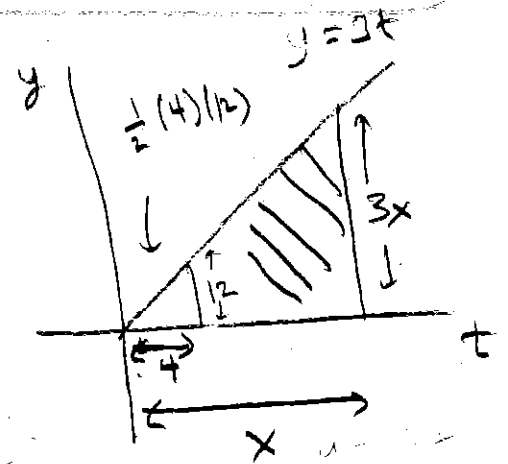
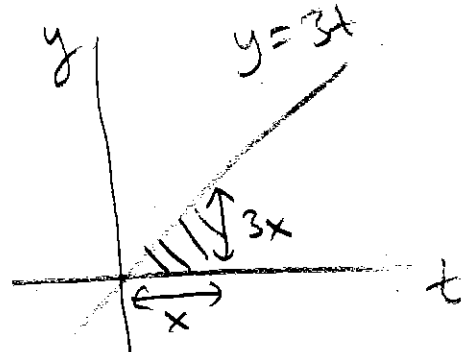
Any observations?

$$g'(x) = 3x$$

What changes if the lower bound a number other than 0? For example, let's try 4?

$$4. h(x) = \int_4^x f(t) dt = \frac{3}{2} x^2 - 24$$

$$h'(x) = 3x$$



# Fundamental Theorem of Calculus

(Part 1): Areas under graphs are antiderivatives!

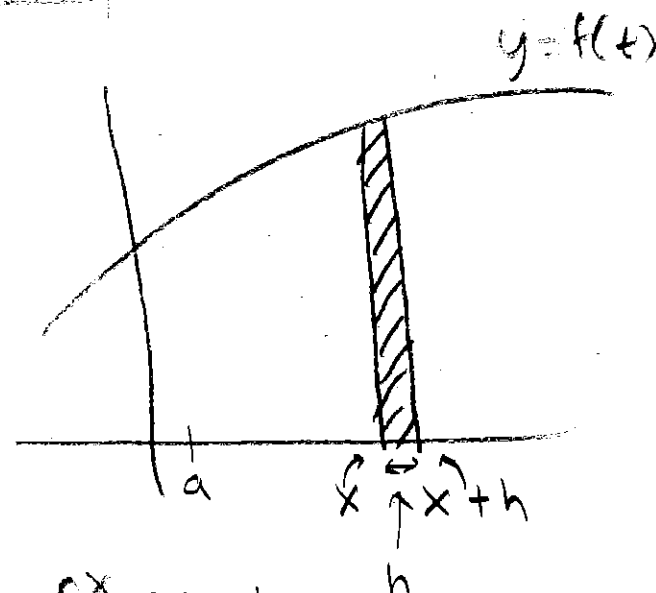
$$\frac{d}{dx} \left( \int_a^x f(t) dt \right) = f(x)$$

That is, for any constant  $a$ , the "accumulated signed area" formula

$$F(x) = \int_a^x f(t) dt$$

is an antiderivative of  $f(x)$ .

ASIDE



$$g(x) = \int_a^x f(t) dt$$

$$g(x+h) - g(x) = \int_x^{x+h} f(t) dt$$

$$\text{Thus, } \frac{g(x+h) - g(x)}{h} = \frac{\int_x^{x+h} f(t) dt}{h}$$

$$\approx f(x)$$

↳ gets closer and closer as  $h \rightarrow 0$

$$\lim_{h \rightarrow 0} \underbrace{\frac{g(x+h) - g(x)}{h}}_{g'(x)} = f(x)$$

## Motivation

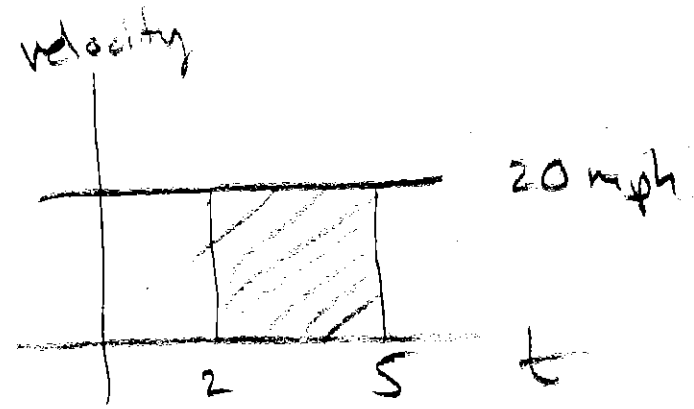
Assume a car is traveling at a constant  $v(t) = 20$  miles/hr.

What do the following represent?

$$1. \int_2^5 \underbrace{v(t)}_{\frac{\text{miles}}{\text{hrs}}} dt = 20(5-2) = 60 \text{ miles}$$

$$2. f(x) = \int_2^x v(t) dt = 20(x-2) = 20x - 60 \text{ MILES}$$

$$3. g(x) = \int_0^x v(t) dt = 20(x-0) = 20x \text{ MILES}$$



DISTANCE  
FUNCTIONS!

Observations?

## Mechanically using FTC (Part 1)

Compute the **derivatives** of the following functions:

$$1. g(x) = \int_3^x \cos(t) dt$$

$$2. h(x) = \int_x^{-2} te^t dt$$

$$3. f(x) = \int_0^{x^3} t + \sin(t) dt$$

$$4. k(x) = \int_{1+x^2}^{x^3} \sqrt{2+t} dt$$

$$\boxed{1} \quad g'(x) = \cos(x)$$

$$\boxed{2} \quad h(x) = - \int_{-2}^x te^t dt$$

$$h'(x) = -x e^x$$

$\boxed{3}$  CHAIN RULE!

$$f'(x) = (x^3 + \sin(x^3)) 3x^2$$

$\boxed{4}$  SPLIT

$$\begin{aligned} k(x) &= \int_{1+x^2}^0 \sqrt{2+t} dt + \int_0^{x^3} \sqrt{2+t} dt \\ &= - \int_0^{1+x^2} \sqrt{2+t} dt + \int_0^{x^3} \sqrt{2+t} dt \end{aligned}$$

$$k'(x) = -\sqrt{2+(1+x^2)} \cdot 2x + \sqrt{2+x^3} \cdot 3x^2$$

## General form of FTOC (Part 1):

$$\frac{d}{dx} \left( \int_{g(x)}^{h(x)} f(t) dt \right) = f(h(x))h'(x) - f(g(x))g'(x)$$

↳ IN OUR  
LAST  
EXAMPLE

$$\frac{d}{dx} \left( \int_{1+x^2}^{x^2} \sqrt{2+t} dt \right) = \sqrt{2+x^2} \cdot (3x^2) - \sqrt{2+(1+x^2)} \cdot (2x)$$

# Fundamental Theorem of Calculus

(Part 2):

If  $F(x)$  any antiderivative of  $f(x)$ ,

$$\int_a^b f(x) dx = F(b) - F(a)$$

ASIDE

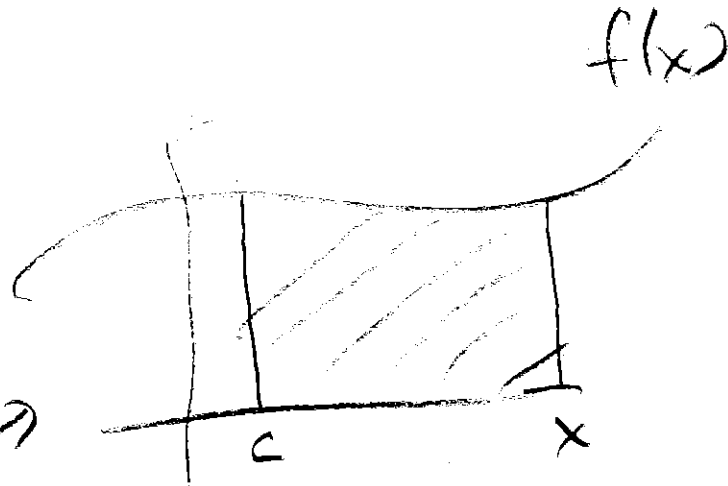
1 IF  $F(x)$  IS AN ANTIDERIVATIVE  
THEN IT REPRESENTS AN ACCUMULATED  
AREA FUNCTION FOR SOME STARTING VALUE  $C$

2 SO  $F(b)$  = "AREA FROM  $c$  TO  $b$ "  
 $F(a)$  = "AREA FROM  $c$  TO  $a$ "

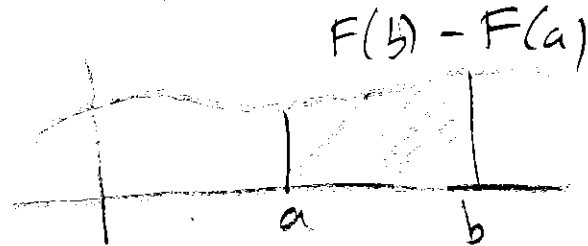
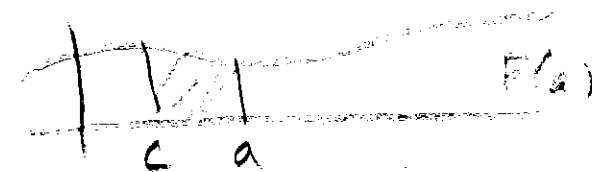
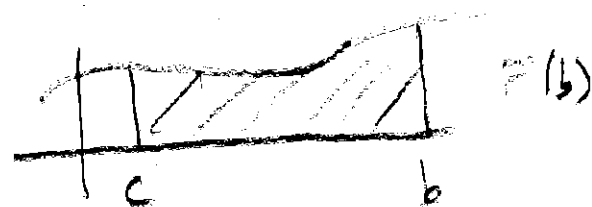
THUS,  $F(b) - F(a)$  = AREA BETWEEN  
 $a$  &  $b$

AND

IT DOESN'T MATTER WHAT  
THE VALUE OF  $C$  WAS!



$$F(x) = \int_c^x f(t) dt$$



## Mechanically using FTOC (Part 2)

Evaluate

$$1. \int_0^1 x^3 dx = \frac{1}{4} x^4 \Big|_0^1 = \frac{1}{4} - 0 = \boxed{\frac{1}{4}}$$

$$F(x) = \frac{1}{4} x^4 + C$$

$$F(b) - F(a)$$

$$F(1) - F(0)$$

$$\left[ \frac{1}{4} (1)^4 + C \right] - \left[ \frac{1}{4} (0)^4 + C \right]$$

$$\frac{1}{4} + C - C$$

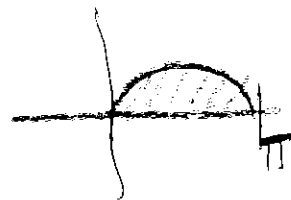
$$\boxed{\frac{1}{4} = 0.25}$$

$$2. \int_0^{\pi} \sin(t) dt = -\cos(t) \Big|_0^{\pi}$$

$$= [-\cos(\pi)] - [-\cos(0)]$$

$$= -(-1) - -1$$

$$= \boxed{2}$$

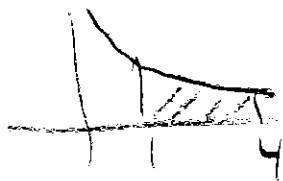


$$3. \int_1^4 \frac{1}{w} dw$$

$$= \ln|w| \Big|_1^4$$

$$= \ln|4| - \ln|1|$$

$$= \boxed{\ln(4)}$$



$$5. \int_1^2 \frac{3}{x^2} dx = 3 \int_1^2 x^{-2} dx$$

$$= 3 \left( \frac{1}{-1} x^{-1} \Big|_1^2 \right)$$

$$= 3 \left( -\frac{1}{x} \Big|_1^2 \right)$$

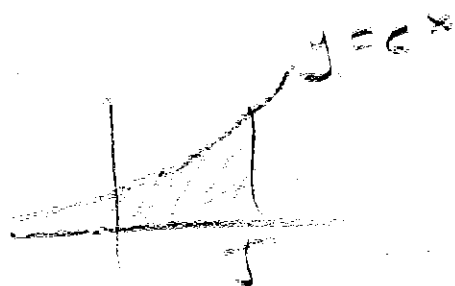
$$= 3 \left( \left[ -\frac{1}{2} \right] - \left[ -\frac{1}{1} \right] \right)$$

$$= \boxed{\frac{3}{2}}$$

$$4. \int_0^5 e^x dx$$

$$= e^x \Big|_0^5$$

$$= e^5 - e^0 = \boxed{e^5 - 1}$$



$$6. \int_1^4 \sqrt{x} dx = \frac{2}{3} x^{3/2} \Big|_1^4$$

$$= \frac{2}{3} \left( \left[ 4^{3/2} \right] - \left[ 1^{3/2} \right] \right)$$

$$= \frac{2}{3} (8 - 1) = \boxed{\frac{14}{3}}$$



$$7. \int_0^1 \frac{1}{1+x^2} dx$$

$$= \tan^{-1}(x) \Big|_0^1$$

$$= [\tan^{-1}(1)] - [\tan^{-1}(0)]$$

$$= \frac{\pi}{4} - 0 = \boxed{\frac{\pi}{4}}$$

$$8. \int_0^{\pi/3} \sec(x)\tan(x) dx$$

$$= \sec(x) \Big|_0^{\pi/3}$$

$$= \sec(\pi/3) - \sec(0)$$

$$= \frac{1}{\cos(\pi/3)} - \frac{1}{\cos(0)}$$

$$= \frac{1}{(1/2)} - \frac{1}{1}$$

$$= 2 - 1 = \boxed{1}$$